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Analytical results for generalized persistence properties of smooth processes

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Abstract. We present a general scheme to calculate within the independent interval approximation generalized (level-dependent) persistence properties for processes having a finite density of zero crossings. Our results are especially relevant for the diffusion equation evolving from random initial conditions—one of the simplest coarsening systems. Exact results are obtained in certain limits, and rely on a new method to deal with constrained multiplicative processes. An excellent agreement of our analytical predictions with direct numerical simulations of the diffusion equation is found.

1. Introduction

Over the past few years, first-passage properties in non-equilibrium situations have attracted much interest, with numerous works devoted to the so-called persistence phenomenon (for a recent review, see [1], and references therein). The simplest setting might be within the field of phase-ordering kinetics where, as domain growth proceeds, the following question naturally arises [2, 3]. What is the fraction $R(t)$ of space which has always been in the same phase (say the + one) up to time t? Equivalently, for zero-temperature quenches, $R(t)$ is the probability that a given spin has never flipped up to time t, and is observed to decay in time as $t^{-\theta}$, where θ is the persistence exponent. The latter can also be considered as a first-passage exponent, since $-\mathrm{d}R(t)/\mathrm{d}t$ represents the probability that an interface first sweeps over a given point at time t. Experimental measurements of θ have been conducted in breath figures [3], and in a system of nematic liquid crystal akin to the two-dimensional (2D) Ising model dynamics [4]. In line with the general view that thermal noise should be asymptotically irrelevant in coarsening systems, several numerical methods [5–8] indicate that θ seems to keep the same value for all quenches in the broken-symmetry region. Unfortunately, exact calculations of this new critical exponent are scarce [9, 10] since, after integrating out at a given location in space the spatial degrees of freedom, one has to deal with an effective stochastic process which is generically non-Markovian. Perturbative techniques to calculate θ have therefore been developed [11–15], especially when a spin can be associated with (the sign of) a Gaussian process, a situation

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common to various closure schemes [16–18] of phase ordering. Other lines of investigation have been concerned with extending the notion of (local) persistence probability. This includes notably the definition of a persistence exponent θ_g for the global order parameter (at T_c [19], and below [6, 7]), the notion of block persistence [6, 7] (which encompasses both θ and θ_g), and two other generalizations [20, 21], relaxing in a different fashion the no spin-flip constraint inherent to the definition of $R(t)$. Here we shall deal with the extension introduced in [20] (see also [22]), which is based on the consideration of the local mean magnetization:

$$M(t) = \frac{1}{t}\int_0^t \mathrm{d}t'\,\sigma(t'). \tag{1.1}$$

This quantity is simply related to the proportion of time spent in one of the two possible phases by a given spin $\sigma \equiv \sigma_r$. The distribution of $M(t)$ also contains the persistence probability, for Prob $\{M(t) = +1\} = R(t)$. Yet, as soon as $|M(t)| < 1$, changes of phase are allowed, and one observes that $M(t)$ converges, when $t \to \infty$, to a broad limiting distribution f_M, reminiscent of the classical arcsin law for Brownian motion, whose singular behaviour $f_M(x) \sim (1-x^2)^{\theta-1}$ at the edges $x = \pm 1$ of its support involves the persistence exponent [20]. Another conceptually related route giving access to θ is to consider, for a fixed value of the level $x < 1$, the persistent large deviations $R(t, x) = \text{Prob}\left\{M(t') \geqslant x,\ \forall t' \leqslant t\right\}$ of the mean magnetization, which can also be interpreted as the (+)-persistence of the process sign$[M(t) - x]$. For large times, this quantity decays as a power law $t^{-\theta(x)}$ [20], and the original persistence exponent appears as the limiting case $\theta = \lim_{x\to 1^-} \theta(x)$ of a continuously varying family of generalized persistence exponents. Let us mention that another infinite hierarchy of exponents appears when one allows a spin to be 'reborn' with a certain probability p each time it flips [21], or when one considers the survival properties of (spatial) domains in coarsening systems [23]. In fact, $R(t, x)$ displays a non-trivial structure even for the simple (uncorrelated) binomial random walk [24].

Although the existence of level-dependent exponents is by no means new (see, e.g., [25] for Gaussian processes, and the more recent work [15]), it is physically very appealing that, as pointed out in [8], both the limit law $f_M(x)$ and the spectra $\theta(x)$ seems to be *universal* for the 2D Ising model in the whole region $T < T_c$, up to a rescaling of x by the corresponding equilibrium value of the magnetization, thus reflecting the fact that coarsening systems, though perpetually out-of-equilibrium, are nevertheless in *local* equilibrium.

We have also argued recently [26] that the global shape of the $\theta(x)$ curve is a sensitive probe of the type of noise (stochastic or deterministic) carried by the motion of the domain walls', thereby suggesting an explanation of the discrepancy currently reported [7] between the persistence exponents of the 2D Ising model and of its continuous counterpart, the time-dependent Ginzburg–Landau (TDGL) equation.

The purpose of this paper is to present a family of models for which generalized persistence properties can be obtained analytically. We shall do this within the framework of the independent interval approximation (IIA) used in [12, 13] to calculate the persistence exponent for a Gaussian random field evolving under the diffusion equation. The IIA has been used later to determine the limit law for this system [20], and our work is a natural continuation of this study. Although the IIA is an uncontrolled approximation, limited to 'smooth' processes (that is, if $\sigma(t) = \text{sign}\,X(t)$, the process $\{X(t)\}_t$ must be everywhere (with respect to t) differentiable), this method has given very accurate results, both for θ and $f_M(x)$. Excellent agreement will also be found here when comparing our results with the spectrum $\theta(x)$ obtained numerically for the diffusion equation. In a certain limit, there will also appear an unexpected similarity with an exact relationship obtained for the $\theta(x)$ exponents of the Lévy-based model studied in [27], although the latter process resembles the one-dimensional (1D) Ising model and is non-smooth in nature. To conclude our motivation, we mention that some of our analytical results rely on a technique to deal with multiplicative random recursions (defining a so-called Kesten variable [28]), on which constraints are enforced. Since such variables show up in a variety of one-dimensional random systems [29, 30], it is also hoped that the method introduced here might prove helpful in tackling new questions in this field.

The paper is organized as follows. In section 2, we review the IIA. Sections 3 and 4 are concerned with the determination of generalized persistence properties when the number of spin-flips is fixed (section 3), and when the time is fixed (section 4, respectively). We conclude in section 5 with a discussion.

2. The independent interval approximation

To fix the notation, we first give a brief account of this method used in [12, 13] to calculate the persistence exponent for the diffusion equation. Owing to dynamic scaling, coarsening systems with algebraic growth laws can be rendered stationary by going over to 'logarithmic' time $\ell = \ln t - \ln t_0$. The principle of the IIA is to assume that the lengths of the (logarithmic) times $l_i = \ln t_i - \ln t_{i-1}$ between successive spin-flips $t_1, t_2, \ldots$ form a renewal process, the common probability distribution function $f(l)$ of these intervals being determined by relating it to the scaling form $\langle\sigma(t_0)\sigma(t)\rangle = a(t_0/t) \equiv A(\ell)$ of the spin–spin autocorrelation function (which is known exactly for the diffusion equation, and more generally for any Gaussian process). The relationship reads, in Laplace space ($\hat{f}(s) \equiv \int_0^\infty \mathrm{d}\ell\, \mathrm{e}^{-s\ell} f(\ell)$),

$$\hat{f}(s) = \frac{1 - \langle l\rangle\,[1 - s\hat{A}(s)]/2}{1 + \langle l\rangle\,[1 - s\hat{A}(s)]/2}. \tag{2.1}$$

In (2.1), the average length $\langle l\rangle$ between two zero crossings is determined by the small-ℓ behaviour $A(\ell) \approx 1 - 2\ell/\langle l\rangle$ of the correlator. If $A(\ell) - 1 \sim \ell^\alpha$ with an exponent $\alpha < 1$, as happens in the 1D Ising model (for which $\alpha = \frac{1}{2}$) or for some interface growth models [31, 32] (where α is related to the roughness exponent), then $A'(0)$ is infinite and the IIA cannot be used as such (see, however, [33]) to enumerate the spin-flips: any change of sign is likely to be followed by a dense sequence of zero crossings (of fractal dimension $D_0 = 1 - \alpha$ [34]). Otherwise, the expected large-time behaviour $R(t) \sim t^{-\theta} \propto \mathrm{e}^{-\theta\ell}$ of the persistence probability can be inferred from the existence of a pole at $s = -\theta$ for $\hat{f}(s)$†. This procedure can be implemented in the Ohta–Jasnow–Kawasaki closure scheme of phase ordering [16] (abbreviated from now on as the 'diffusion equation'), where the dynamics of a spin is recast in terms of an auxiliary diffusing field,

$$\frac{\partial}{\partial t}\phi(\boldsymbol{r}, t) = D\nabla^2\phi(\boldsymbol{r}, t) \tag{2.2}$$

evolving from zero-mean random (Gaussian, or simply short-ranged) initial conditions. The zeros of this diffusing field are interpreted as representing the positions of domain walls: $\sigma_{\boldsymbol{r}}(t) = \operatorname{sign}\phi(\boldsymbol{r}, t)$. At a given location in space, the normalized process $X(t) \equiv \phi(\boldsymbol{r}, t)/\left[\left\langle\phi^2(\boldsymbol{r}, t)\right\rangle\right]^{1/2}$ is still Gaussian, therefore solely determined by its two-time correlation function:

$$c(t_0, t) = \langle X(t_0)X(t)\rangle = \left[\frac{4t_0 t}{(t_0 + t)^2}\right]^{d/4} \tag{2.3}$$

† Actually, the persistence probability is slightly different from the distribution function of intervals, since in the former, one does not necessarily have zeros at both endpoints of the interval. However, the large-time behaviour ($\propto \exp(-\theta\ell)$) of these two quantities is the same.

which depends only on the ratio t_0/t of the two times. The spin–spin autocorrelation function is $\langle\sigma(t_0)\sigma(t)\rangle = (2/\pi)\arcsin c(t_0, t)$ (this equality holds true for any Gaussian process), and reads, when expressed in terms of the stationary timescale $\ell = \ln t - \ln t_0$,

$$A(\ell) = \frac{2}{\pi}\arcsin\left\{[\cosh(\ell/2)]^{-d/2}\right\}. \tag{2.4}$$

Plugging this back into (2.1) gives values of the persistence exponent θ which are extremely close to those obtained by a direct simulation of (2.2): one finds, respectively, in $d = 1, 2, 3$ space dimensions $\theta_{IIA} = 0.1203$, 0.1862, 0.2358, while $\theta_{num} = 0.1207(5)$, 0.1875(10), 0.2380(15) [12, 13].

The IIA can also be used to calculate the limit law of the mean magnetization [20]. For our later purposes, it is convenient to introduce the ratios $z_i = t_{i-1}/t_i = \mathrm{e}^{-l_i}$ of two successive flip times. We denote their common distribution by f_Z. The Laplace transform $\hat{f}(s)$ of the probability distribution of the l_is is nothing but the Mellin transform $\int_0^1 \mathrm{d}z\, z^{s-1} f_Z(z)$ of that of the z_is, which implies that

$$f_Z(z) \approx a_0 z^{\theta-1} \qquad z \to 0. \tag{2.5}$$

In (2.5), the proportionality constant a_0 can be computed from the residue at $s = -\theta$ of the Mellin–Laplace transform (2.1)). According to the sign of $\sigma(t)$, one can now rewrite $tM(t) = \pm[t_1 - (t_2 - t_1) + \cdots]$ as

$$M(t) = \pm\left(1 - 2\mathrm{e}^{-\xi} W_n\right). \tag{2.6}$$

Here, $\xi = \ln t - \ln t_n$ is the length of (logarithmic) time since the last zero crossing, and the 'weights' W_ns (which belong to the interval (0, 1)) are identified as Kesten variables, as they obey the following random multiplicative relation:

$$W_n = 1 - z_n W_{n-1}. \tag{2.7}$$

In the large-time limit, the random variable W_n converges in distribution to the solution of $W = 1 - ZW$, which means that the probability distribution g of W has to satisfy the following integral equation:

$$g(w) = \int_{1-w}^{1} \mathrm{d}w' w'^{-1} g(w') f_Z\left(\frac{1-w}{w'}\right). \tag{2.8}$$

The solution of (2.8) represents the main step in the determination of the limit law of the mean magnetization, since for a renewal process the law of the backward recurrence time ξ is known: $\left\langle \mathrm{e}^{-s\xi}\right\rangle = (1 - \hat{f}(s))/\langle l\rangle s$. One of the very few cases where the Dyson–Schmidt integral equation (2.8) can be inverted explicitly is when the distribution f_Z is a pure power law in z, that is when $f_Z(z) = \theta z^{\theta-1}$, $\forall z \in (0, 1)$ [30]. One then finds

$$g(w) = B^{-1}(\theta, \theta+1)(1-w)^{\theta-1} w^{\theta} \tag{2.9}$$

($B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ being the usual Beta function), and eventually that the limit density of the mean magnetization is a Beta law on $(-1, 1)$:

$$f_M(x) = B^{-1}\left(\tfrac{1}{2}, \theta\right)(1 - x^2)^{\theta-1}. \tag{2.10}$$

This particular choice for f_Z, proportional to the tail behaviour of the true distribution, nevertheless contains much of the physics of the persistence phenomenon since, in all coarsening systems, the limit density of the mean magnetization is found to be numerically indistinguishable on its whole support from the law (2.10) [8, 20, 22, 26]. This can also be demonstrated within the IIA for the diffusion equation (using the full law $f_Z(z)$ determined by (2.1)–(2.4)), where one finds (by extrapolating the large-order behaviour of the moments of M) that the tail behaviour $f_M(x) \sim (1 - x^2)^{\theta-1}$ contains an overwhelming proportion of the full measure [20].

Let us now define the quantities of interest in this paper. Depending on whether the number of spin-flips n or the time t is fixed, we want to calculate

$$R(n, x) = \text{Prob}\,\{M_k \geqslant x,\ \forall\, k = 1, 2, \ldots, n\} \tag{2.11}$$

$$R(t, x) = \text{Prob}\,\big\{M(t') \geqslant x,\ \forall\, t' \leqslant t\big\}. \tag{2.12}$$

In (2.11), $M_k = M(t_k)$ is the value of the mean magnetization at exactly the kth flip time. These two aspects of generalized persistence properties are different, but we shall show that they are related.

3. Persistent large deviations of the mean magnetization when the number of spin-flips is fixed

3.1. General formalism

Let us start with a study of $R(n, x)$. We assume from now on that initially $\sigma(0) = +1$, and that the first spin-flip ratio z_1 possesses the same law as the others: $z_1 = t_0/t_1$, with $t_0 = 1$. Since the last condition $\{M_n \geqslant x\}$ in (2.11) can only be violated—had it been satisfied earlier—when a spin is journeying in the minus phase, that is, after an odd number of spin-flips, one has $R(2n + 1, x) = R(2n, x)$. Expressed in terms of the weights W_{2k} introduced earlier, the n conditions to be maintained at the $2n$th spin-flip time are

$$R(2n, x) = \text{Prob}\left\{W_2 < \frac{1-x}{2}, W_4 < \frac{1-x}{2}, \ldots, W_{2n} < \frac{1-x}{2}\right\}. \tag{3.1}$$

Instead of considering all the instances taken by the W_{2k}s and then applying the constraints (3.1), it is sufficient to consider the ensemble of constrained variables $W_{2k}^{(x)} \in [0, \frac{1-x}{2}]$ transformed by $W_{2k}^{(x)} = 1 - z_{2k}(1 - z_{2k-1} W_{2k-2}^{(x)})$ (henceforth to simplify the notation we do not display the dependence of the variables W_{2k} upon x). This does not define a map on $[0, \frac{1-x}{2}]$, since some variables may fall outside the interval. Let us introduce $\rho_{2k}(x)$, which represents the fraction of the variables which are left on $[0, \frac{1-x}{2}]$ during iteration $2k$. Therefore, one has,

$$R(2n, x) = \prod_{k=1}^{n} \rho_{2k}(x). \tag{3.2}$$

The key to the argument is now to write the dynamics for the distributions of the constrained variables W_{2k} on the interval $[0, \frac{1-x}{2}]$. Their renormalized densities h_{2k} ($\int_0^{(1-x)/2} \mathrm{d}w\, h_{2k}(w) = 1$) now have to obey

$$\rho_{2k}(x) h_{2k}(w) = \int_{1-w}^{1} \frac{\mathrm{d}w'}{w'} f_Z\left(\frac{1-w}{w'}\right) \int_{1-w'}^{1} \frac{\mathrm{d}w''}{w''} h_{2k-2}(w'') f_Z\left(\frac{1-w'}{w''}\right). \tag{3.3}$$

The subsequent step is to recognize that in the large-n limit the renormalized densities h_{2n}s have to reach a stationary distribution $h(w)$, vanishing identically for $w \geqslant \frac{1-x}{2}$. Once this stationary regime is reached, one has simply

$$R(2n, x) \sim [\rho(x)]^n \qquad n \gg 1 \tag{3.4}$$

with $\rho(x) = \lim_{n\to\infty} \rho_{2n}(x)$ determined self-consistently through the solution of

$$\rho(x)h(w) = \int_{1-w}^{1} \frac{\mathrm{d}w'}{w'} f_Z\left(\frac{1-w}{w'}\right) \int_{1-w'}^{(1-x)/2} \frac{\mathrm{d}w''}{w''} h(w'') f_Z\left(\frac{1-w'}{w''}\right) \tag{3.5}$$

subject to the constraint that $h(w)$ is a normalized probability distribution on the interval $[0, \frac{1-x}{2}]$. We have to stress that this exponential decay of $R(n, x)$ is in agreement with the expected algebraic behaviour of $R(t, x)$ since, for a smooth process, the typical number of spin-flips up to time t scales as $\ln t/\langle l\rangle$. This would no longer be true for the 1D-Ising model (or for the Lévy-based model studied in [27]), where the typical number of spin-flips behaves rather as $t^{1/2}$ [35] (or t^θ for the Lévy model), and $R(n, x)$ instead goes to zero algebraically.

3.2. Exact solution when $f_Z(z) = \theta z^{\theta-1}$

Since the determination of the unconstrained probability density g of the Ws is already a difficult problem, there is no hope of solving (3.5) for an arbitrary distribution f_Z. Nevertheless, when $f_Z(z) = \theta z^{\theta-1}$, the structure of (3.5) still allows for an exact calculation of $\rho(x)$, which we now present. (Note that for the probability distribution function $f(l)$ of intervals between spin-flips on the logarithmic timescale, this choice for $f_Z(z)$ corresponds to a Poisson process: $f(l) = \theta \mathrm{e}^{-\theta l}$.) In this case, rewriting (3.5) as

$$(1-w)^{1-\theta} h(w) = \frac{\theta^2}{\rho(x)} \int_{1-w}^{1} \frac{\mathrm{d}w'}{w'^\theta} (1-w')^{\theta-1} \int_{1-w'}^{(1-x)/2} \frac{\mathrm{d}w''}{w''^\theta} h(w'') \tag{3.6}$$

two successive differentiations with respect to w show that $h(w)$ obeys the following (purely local) differential equation:

$$w(1-w)\frac{\mathrm{d}^2 h}{\mathrm{d}w^2} + [1-\theta-(3-2\theta)w]\frac{\mathrm{d}h}{\mathrm{d}w} - \left[(\theta-1)^2 - \theta^2/\rho(x)\right] h = 0 \tag{3.7}$$

which is recognized as Gauss' hypergeometric differential equation. The integral equation (3.6) demands that $h(0) = 0$, and this selects the solution

$$h(w) = \text{norm}\, w^\theta {}_2F_1\left(1 - \frac{\theta}{\sqrt{\rho(x)}}, 1 + \frac{\theta}{\sqrt{\rho(x)}}; 1+\theta; w\right) \tag{3.8}$$

where ${}_2F_1(a, b; c; w)$ is the hypergeometric series:

$${}_2F_1(a, b; c; w) \equiv \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{w^n}{n!}. \tag{3.9}$$

(Note that for $x = -1$, $\rho(-1) = 1$, (3.8) does reduce to (2.9), since ${}_2F_1(1-\theta, 1+\theta; 1+\theta; w) = (1-w)^{\theta-1}$.) To determine $\rho(x)$, we need a second boundary condition, which is obtained by differentiating the left-hand side of (3.6) with respect to w, whereupon we set $w = \frac{1-x}{2}$. Then the right-hand side of the obtained equation is identically zero, since the range of the remaining integration vanishes. Using the formula

$$\frac{\mathrm{d}}{\mathrm{d}w}\left[(1-w)^{a+b-c} w^{c-1} {}_2F_1(a, b; c; w)\right] = -(1-c)(1-w)^{a+b+c-1} w^{c-2} \times {}_2F_1(a-1, b-1; c-1; w) \tag{3.10}$$

we obtain, for $\theta \neq 1$,

$${}_2F_1\left(\frac{-\theta}{\sqrt{\rho(x)}}, \frac{\theta}{\sqrt{\rho(x)}}; \theta; \frac{1-x}{2}\right) = 0. \tag{3.11}$$

(For $\theta = 1$, equation (3.10) is not valid, but a similar identity applies, yielding ${}_2F_1(1 - \rho^{-1/2}(x), 1+\rho^{-1/2}(x); 2; \frac{1-x}{2}) = 0$.) Equation (3.11) determines implicitly $\rho(x)$ as a function of x, for a given value of θ. This formula can be inverted when $\theta = \frac{1}{2}$, to give

$$\rho(x) = \left[\frac{2}{\pi}\arcsin\sqrt{\frac{1-x}{2}}\right]^2 \qquad \left(\theta = \tfrac{1}{2}\right). \tag{3.12}$$

In this case, the constrained density $h(w)$ also has an explicit expression:

$$h(w) = \text{norm}\,\frac{2\arcsin\sqrt{(1-x)/2}}{\pi\sqrt{1-w}}\,\sin\left(\frac{\pi\arcsin\sqrt{w}}{2\arcsin\sqrt{(1-x)/2}}\right) \qquad \left(\theta = \tfrac{1}{2}\right) \tag{3.13}$$

where the normalization factor (which depends of course on x) can also be calculated. For other values of θ, equation (3.11) is generically multivalued, but there is a unique determination of $\rho(x)$ in the physical range (0, 1), and the ensuing solution can be obtained numerically with an arbitrary precision.

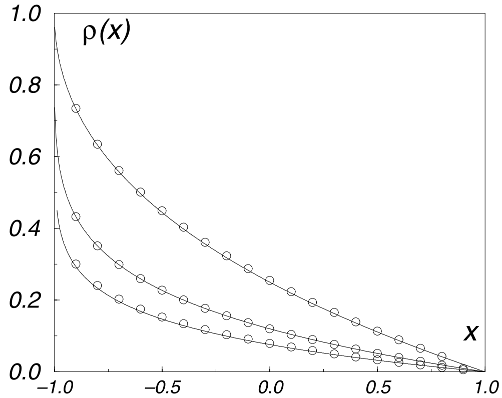

Figure 1. The decay rate $\rho(x)$ of $R(2n, x)$ (as defined by (3.4)) for three representative values of θ: $\theta = 0.121$ (the value of the persistence exponent for the 1D diffusion equation), $\theta = 0.2$ (that of the 2D TDGL equation [7, 26]), and $\theta = 0.5$, from bottom to top. Full curve, exact formula (3.11); circles, numerical simulations.

Figure 1 shows our analytical prediction (3.11) for three different values of θ, compared with the results of a direct numerical simulation of the process defined by (2.7)–(3.1), the z_ns being distributed according to $f_Z(z) = \theta z^{\theta-1}$. The agreement found is excellent. Numerical measurements of the expected exponential decay of $R(n, x)$ for the diffusion equation are hard to perform, for the number of spin-flips scales as $\ln t/\langle l\rangle$, which is only of the order of 2–3 even for times $t = 10^7$.

Even if (3.11) is rather unwieldy for a generic value of θ, one can extract from it the behaviour of $\rho(x)$ at the two edges $x = \pm 1$ of the spectrum. This turns out to also be possible for a generic law f_Z, and we shall now sketch these calculations.

3.3. Limiting behaviour of $\rho(x)$ when $x \to 1$

When $x \to 1$, $\rho(x) \to 0$, and (3.11), (3.9) show that the proper scaling variable is $S = (1 - x)/2\rho(x)$. In this scaling limit, we have to keep all terms in the hypergeometric series (3.8), which simplifies to

$$0 = \sum_{n\geqslant 0} \frac{\Gamma(\theta)}{\Gamma(\theta+n)} \frac{(-S)^n}{n!} = \Gamma(\theta) S^{(1-\theta)/2} J_{\theta-1}(2\sqrt{S}) \tag{3.14}$$

where $J_{\theta-1}(v)$ is a Bessel function. The solution of (3.14) is therefore

$$\rho(x) \approx \left(\frac{2\theta}{j_{\theta-1,1}}\right)^2 \frac{1-x}{2} \tag{3.15}$$

where $j_{\theta-1,1}$ is the first strictly positive zero of the Bessel function $J_{\theta-1}(v)$. (One can check with (3.12) that this result is correct for $\theta = \frac{1}{2}$, since in this case one has simply $J_{-1/2}(v) = \cos v$.) As expected, the decay rate $-\ln \rho(x)$ of the generalized persistence probability $R(n, x)$ diverges when $x \to 1$, since no spin-flips are allowed in this limit.

It is also possible to estimate the behaviour of $\rho(x)$ when $x \to 1$ for the 'true' law f_Z given by the inverse transform of (2.1). To do this, we need to determine the behaviour of $f_Z(z)$ near $z = 1$. This can be done by developing (2.1) for $s \to \infty$ in powers of $1/s$. The resulting expression involves the coefficients of the Taylor series expansion of the field-correlator near $\ell = 0$: $C(\ell) = \langle X(0)X(\ell)\rangle = 1 + c_2\ell^2/2! + c_4\ell^4/4! + \cdots$. To lowest order in $z \to 1$, the result for the diffusion equation in d dimensions is

$$f_Z(z) \approx a_1(1-z) \qquad \text{with} \quad a_1 = \frac{c_2^2 - c_4}{2c_2} = \frac{d+2}{32}. \tag{3.16}$$

A careful study of the integral equation (3.5), using the fact that the singular part of $g(w)$ in the limits $w \to 0, 1$ is still given by (2.9) for any law $f_Z(z) \approx a_0 z^{\theta-1}$, shows eventually that $\rho(x) \sim (1-x)^2$ when $x \to 1$.

3.4. Limiting behaviour of $\rho(x)$ when $x \to -1$

In the opposite limit $x \to -1$, $\rho(x) \to 1$, and one can derive, by retaining the first two terms in the series (3.11) (the natural expansion parameters being $1 - \sqrt{\rho(x)}$ and $y \equiv (1+x)/2$), the following expression:

$$1 - \sqrt{\rho(x)} \approx y^\theta \left[\frac{\theta\Gamma^2(\theta)}{\Gamma(2\theta)} - y\frac{\theta^3\Gamma^2(\theta)}{(1-\theta)\Gamma(2\theta)} - y^\theta(1 + \theta\pi\cot\theta\pi)\right]^{-1} \tag{3.17}$$

$$\approx \frac{1}{2} B^{-1}(\theta, \theta+1) \left(\frac{1+x}{2}\right)^\theta. \tag{3.18}$$

The higher-order correction term in (3.17), even though it looks superficially divergent for $\theta = 1$, can be analytically continued to give the correct result

$$1 - \sqrt{\rho(x)} \approx \frac{(1+x)/2}{1 - (1+x)/2 - (1+x)/2 \ln((1+x)/2)} \qquad x \to 1 \quad \theta = 1. \tag{3.19}$$

To lowest non-trivial order in $(1+x)/2$, using (2.9), equation (3.18) is therefore equivalent to

$$\rho(x) \approx 1 - B^{-1}(\theta, \theta+1) \left(\frac{1+x}{2}\right)^\theta \approx \int_0^{(1-x)/2} \mathrm{d}w\, g(w) \qquad x \to -1. \tag{3.20}$$

This result has a simple and intuitive interpretation: in this weakly constrained limit, the W_{2k}s obeying (3.1) are simply asymptotically distributed according to the *unconstrained* probability distribution $g(w)$.

Again, such a behaviour is not restricted to the particular choice $f_Z(z) = \theta z^{\theta-1}$: for an arbitrary distribution behaving as in (2.5), one can show, still using the fact that $g(w) \sim w^\theta (1-w)^{\theta-1}$ when $w \to 0, 1$, that the ansatz

$$1 - \rho(x) \sim \left(\frac{1+x}{2}\right)^\theta \qquad x \to -1 \tag{3.21}$$

does solve the integral equation (3.5).

4. Persistent large deviations of the mean magnetization when the time is fixed

Even for the simplest choice $f_Z(z) = \theta z^{\theta-1}$ (corresponding to a Poisson process with mean $1/\theta$ on the logarithmic timescale), we have not succeeded in calculating exactly the complete spectra $\theta(x)$ of generalized persistence exponents: when expressed in terms of the spin-flip ratios, the constraints that one has to enforce are nonlinear and non-local. For an arbitrary law f_Z, we can nevertheless offer a *systematic* perturbative development in the neighbourhood of $x = 1$, and a non-perturbative resummation in the vicinity of $x = -1$, the latter being presumably exact (though we cannot strictly prove it).

4.1. Limiting behaviour of $\theta(x)$ when $x \to 1$

The first idea is to follow by continuity $\theta(x)$ from $\theta = \theta(1)$ in an expansion in the number of spin-flips which have occurred up to time t. The first term of the expansion for $R(t, x)$ is trivial, and just corresponds to the persistence probability. That is, if no spin-flips have occurred up to time t, then the first flip time $t_1 = 1/z_1 \geqslant t$, and the probability of this event is $R_0 = \int_0^{1/t} \mathrm{d}z_1\, f_Z(z_1)$. If just one spin-flip takes place up to time t, then the two first flip times satisfy $t_1 < t \leqslant t_2$, or equivalently the two first flip ratios obey $t^{-1} < z_1 \leqslant 1$, $0 \leqslant z_2 \leqslant (tz_1)^{-1}$. The magnetization $M(t) = t^{-1}[t_1 - (t - t_1)] = -1 + 2t_1/t$ has to satisfy $M(t) \geqslant x$ to contribute to $R(t, x)$. This tightens the upper bound on the first flip-time ratio z_1, and gives a contribution

$$R_1 = \int_{1/t}^{1/(yt)} \mathrm{d}z_1\, f_Z(z_1) \int_0^{(tz_1)^{-1}} \mathrm{d}z_2\, f_Z(z_2) \tag{4.1}$$

to $R(t, x)$ (as in section 3, we still denote $y = (1 + x)/2$; note also that the IIA has been made by writing the joint probability distribution of z_1, z_2 in factorized form). Consider now the situation where a spin has experienced two spin-flips up to time t. This corresponds to the following bounds on z_1, z_2, z_3:

$$t^{-1} < z_1 \leqslant 1 \qquad (tz_1)^{-1} < z_2 \leqslant 1 \qquad 0 \leqslant z_3 \leqslant (tz_1z_2)^{-1}. \tag{4.2}$$

There are now two contributions to $R(t, x)$ to distinguish, according to the range of values taken by the first flip-time ratio z_1. If the latter obeys the same bounds as in (4.1), then, irrespectively of the subsequent flip-time sequence z_2, z_3, it is not hard to check that such an instance will always contribute to $R(t, x)$, and the bounds on z_2, z_3 are just those defined in (4.2):

$$R_{2,1} = \int_{1/t}^{1/(yt)} \mathrm{d}z_1\, f_Z(z_1) \int_{(tz_1)^{-1}}^1 \mathrm{d}z_2\, f_Z(z_2) \int_0^{(tz_1z_2)^{-1}} \mathrm{d}z_3\, f_Z(z_3). \tag{4.3}$$

The remaining contribution to $R(t, x)$ after two spin-flips reads

$$R_{2,2} = \int_{1/(yt)}^1 \mathrm{d}z_1\, f_Z(z_1) \int_y^1 \mathrm{d}z_2\, f_Z(z_2) \int_0^{(tz_1z_2)^{-1}} \mathrm{d}z_3\, f_Z(z_3). \tag{4.4}$$

We now evaluate the sum of these four contributions in the long-time limit, *and* for $y = (1 + x)/2$ close to 1. The derivation will show that we do not need to take into account higher-order contributions in the number of spin-flips if we content ourselves with an expansion of $\theta(x) - \theta$ to lowest non-trivial order in $(1 - x)/2 = 1 - y \to 0$. In most of the integrals, the large-t behaviour is determined by the small-z form (2.5) of the distribution function of the ratio of flip times, while other spin-flip ratios can be summed over without restriction due to the fact that $y \to 1$. For instance, to lowest-order, equation (4.1) simplifies to

$$R_1 \approx a_0 \int_{1/t}^{1/(yt)} \mathrm{d}z_1\, z_1{}^{\theta-1} \int_0^1 \mathrm{d}z_2\, f_Z(z_2) \approx \frac{a_0}{\theta t^\theta}(y^{-\theta} - 1) \approx \frac{a_0}{t^\theta}(1 - y). \tag{4.5}$$

This kind of simplification is also at work for R_0 and $R_{2,1}$, which are also found to be simply proportional to $t^{-\theta}$ with y-dependent amplitudes, but not for $R_{2,2}$, where the intermediate integral in equation (4.4) shows that what matters is also the behaviour (3.16) of $f_Z(z)$ when $z \to 1$. One eventually finds that

$$R_{2,2} \approx \frac{a_0{}^2 a_1}{2\theta} \frac{(1 - y)^2 \ln t}{t^\theta}. \tag{4.6}$$

The appearance of a term involving a logarithm of the time is mandatory to determine $\theta(x)$, since if we develop the expected algebraic behaviour $R(t, x) \sim t^{-\theta(x)}$ for both $t \gg 1$ and $x \to 1$, we would obtain

$$\begin{aligned} t^{-\theta(x)} &\approx \exp\left[-\theta(1)\ln t + (1 - x)\theta'(1)\ln t + \cdots\right] \\ &\approx t^{-\theta}\left[1 - (1 - x)\theta'(1)\ln t + \cdots\right]. \end{aligned} \tag{4.7}$$

Comparing the two expansions (4.6) and (4.7), one finds that for $x \to 1$

$$\theta(x) \approx \theta\left[1 - \frac{a_0 a_1}{2\theta}\left(\frac{1 - x}{2}\right)^2\right] \tag{4.8}$$

where a_0 and a_1 are the coefficients describing the behaviour of $f_Z(z)$ near $z = 0$ and 1, respectively (equations (2.5) and (3.16)).

4.2. Limiting behaviour of $\theta(x)$ when $x \to -1$

The preceding subsection has shown that there is an interplay between the distributions of the flip-time ratios, the level x, and the time t, which appears to be hard to handle in a systematic fashion. More precisely, the distribution of the number $N_t^{(x)}$ of spin-flips which have occurred up to time t and which contribute to the generalized persistence probability $R(t, x)$, depends on the level x. A scaling argument such as $\langle N_t^{(x)}\rangle \sim \langle N_t^{(-1)}\rangle = \ln t/\langle l\rangle$—although qualitatively correct to explain the exponential dependence of $R(n, x)$ and the converse algebraic one of $R(t, x)$—does not suffice for a quantitative calculation. However, we expect that for $x \to -1$ we can use the unconstrained distribution for the number $N_t \equiv N_t^{(-1)}$ of spin-flips which have taken place up to time t, providing that we take into account the correct form of the survival probability $R(2n, x)$ at every other time step. Namely, one should have the decoupling equation:

$$\mathrm{Prob}\left\{M(t') \geqslant x,\ \forall t' \leqslant t\right\} \approx \mathrm{Prob}\left\{M(t_k) \geqslant x,\ \forall k \leqslant n\right\} \times \mathrm{Prob}\left\{N_t = n\right\}. \tag{4.9}$$

Taking equation (4.9) for granted, one can express the first-passage probability at the level x at time t as

$$-\frac{\mathrm{d}}{\mathrm{d}t} R(t, x) = \sum_{n=0}^{\infty} R(2n, x)\,\mathrm{Prob}\left\{N_t = 2n + 1\right\}. \tag{4.10}$$

Using the fact that the Laplace transform ($s \leftrightarrow \ell = \ln t$) of Prob $\{N_t = n\}$ is equal to $\hat{f}^n(s)[1 - \hat{f}(s))]/s$, and that the asymptotic form of the generalized persistence probability after a fixed number of spin-flips is $R(2n, x) \sim [\rho(x)]^n$, equation (4.10) boils down to a geometric summation in Laplace space. A non-trivial pole emerges at $s = -(\theta(x) + 1)$, where $\theta(x)$ obeys

$$\hat{f}(-\theta(x)) = \frac{1}{\sqrt{\rho(x)}}. \tag{4.11}$$

It is interesting to note that (4.11) is exactly the equation derived (also within the IIA) in [21] for the exponent $\tilde{\theta}(p)$ associated with the survival probability of a spin which is reborn with probability p each time it flips, providing that we make the correspondence $p = \sqrt{\rho(x)}$. There is, however, a difference between this extension of the notion of persistence and the one we are studying here. In our case, the survival probability $\rho(x)$ imposed every other time step is not an external parameter, but is directly inherited from the dynamics. Nevertheless, one expects that the difference between these two aspects should vanish when the survival probability (be it either p or $\sqrt{\rho(x)}$) is close to 1, thus constituting another plausible argument backing (4.9)†.

Now, as $\theta(x) \to 0$ when $x \to -1$, the expansion of the Mellin–Laplace transform $\hat{f}(s) \approx \hat{f}(0) + s\hat{f}'(0) \equiv 1 - s\,\langle l\rangle$ for a small argument $s = -\theta(x)$ gives

$$\theta(x) \approx \langle l\rangle^{-1}\left(1 - \sqrt{\rho(x)}\right). \tag{4.12}$$

For the case $f_Z(z) = \theta z^{\theta-1}$, where we have an exact expression for $\rho(x)$, the coefficients recombine neatly (using in turn (3.18) and (2.10)) to give

$$\theta(x) \approx \frac{\theta}{2}B^{-1}(\theta, \theta+1)\left(\frac{1+x}{2}\right)^{\theta} \approx \theta\int_{-1}^{x} \mathrm{d}x'\, f_M(x') \tag{4.13}$$

an equality valid again to lowest order in $x \to -1$. This seems to be the only case where this relationship, already found in [27], holds true (thus correcting a slight lack of imprecision in [26]). Otherwise, using (3.21), one has simply an asymptotic proportionality:

$$\theta(x) \sim (1+x)^{\theta} \sim \int_{-1}^{x} \mathrm{d}x'\, f_M(x')\; x \to -1. \tag{4.14}$$

With this proviso, we believe that, in the limit $x \to -1$, the relationship (4.14) between the spectrum of generalized persistence exponents and the limit law of the mean magnetization should hold for *any* coarsening system, independently of the underlying nature (smooth or non-smooth) of the process. Indeed, sites satisfying (2.12) must at least have their local magnetization $M > x$, and in the long-time limit there is a fraction $\int_x^1 \mathrm{d}x'\, f_M(x')$ of such sites. Conversely, the decay rate $\theta(x)$ (with respect to the logarithmic timescale $\ell = \ln t$) of the generalized persistence probability $R(t, x)$ should be proportional to $\int_{-1}^x \mathrm{d}x'\, f_M(x')$, at least when correlations and constraints can be neglected, that is for $x \to -1$. The simplest possibility (on dimensional grounds) is then that the proportionality factor between $\theta(x)$ and $\int_{-1}^x \mathrm{d}x'\, f_M(x')$ should be given by the usual persistence exponent θ. However, as testified by

† Indeed, while this paper was being accepted for publication, two very recent works [36] dealing with another smooth Gaussian process for which our method is relevant have been brought to our attention. These authors have calculated exactly (independently and by completely different methods) the partial survival exponent $\tilde{\theta}(p) = \frac{1}{4} - \frac{3}{2\pi}\arcsin\frac{p}{2}$ for a randomly accelerated particle (i.e. for which the position x_t obeys $\mathrm{d}^2x_t/\mathrm{d}t^2 = \eta_t$, with η_t being a Gaussian white noise). This stochastic process (see [1] for a list of references on this problem), being the integral of Brownian motion, has a finite density of zero crossings $1/\langle l\rangle = \sqrt{3}/(2\pi)$. The fact that our equation (4.12) does match their result $\tilde{\theta}(p) \approx (\sqrt{3}/2\pi)(1-p)$ in the $p \to 1$ limit (identifying p with $\sqrt{\rho(x)}$ in this weakly constrained regime, as we have argued before) gives further credence to the general validity of the results we have obtained in this section.

(4.12), other constants may show up when θ does not completely parametrize the distribution of intervals between zero crossings (i.e. when the latter is not exactly equal to $f(l) = \theta e^{-\theta l}$, which implies in particular that $\langle l \rangle$ and $1/\theta$ are different), which is the generic situation for smooth processes. Note also that, for the Lévy-model studied in [27], there is no other scale than that given by θ (when the Lévy laws have an infinite mean, i.e. for an exponent $\theta < 1$), and—thanks to the Sparre Andersen theorem (see, for example, the appendix of [24])—the equality $\theta(x) = \theta \int_{-1}^{x} dx' f_M(x')$ holds exactly for any level x.

4.3. Comparison with spectra measured for the diffusion equation

We now turn to a comparison of our analytical results with numerical measurements of $\theta(x)$ conducted for the diffusion equation (figure 2), in one and two space dimensions.

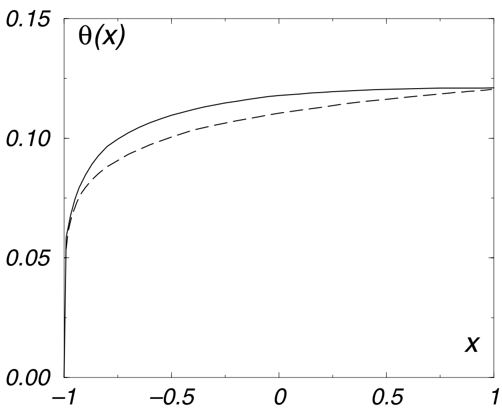

Figure 2. $\theta(x)$ as measured numerically for the 1D diffusion equation (upper, full curve, system size 2×10^7), and for a process with uncorrelated ratios of flip times, power law distributed according to $f_Z(z) = \theta z^{\theta-1}$, with $\theta = 0.121$ (lower, broken curve). 500 values for x are considered in both cases.

The behaviours predicted by (4.8)–(4.14) are confirmed, even though they have been derived within the IIA. At the lower edge of the spectrum, a power-law rise $\theta(x) \sim (1+x)^\theta$ is compatible with the data (despite the obvious numerical imprecision in this region). At the upper edge of the spectrum, where we have a much smaller statistical uncertainty, $\theta(x)$ approaches its limiting value $\theta \approx 0.121(1)$ with a vanishing slope. A numerical fit of the data of the form $\theta(x) - \theta \sim (1-x)^\mu$ would give an exponent $\mu \approx 2.2 \pm 0.1$, which is in fair agreement with our prediction (4.8). We have also represented in figure 2 the spectrum of generalized persistence exponents obtained by direct simulation of a temporal process for which the laws of successive spin-flip ratios are taken to be uncorrelated and distributed according to a pure power law $f_Z(z) = \theta z^{\theta-1}$, with an exponent $\theta = 0.121$. For such a zero-dimensional process, clean power laws extending over 15 decades of time can be obtained without much effort, and the numerical accuracy on the $\theta(x)$ spectrum is very high. Let us note that in this case the probability density function $f_Z(z)$ does not vanish near $z = 1$ and, browsing through the chain of arguments leading to (4.8), the correct prediction near $x \to 1$ is that $\theta(x)$ reaches its limiting value with a finite slope. This is indeed observed in our simulations. One also remarks that, even within this 'doubly' simplified model, the overall agreement with the real spectra is rather good.

To quantify better the effect of temporal correlations, we have also determined numerically the $\theta(x)$ spectrum for a process with a distribution of (uncorrelated) intervals given by the full law (2.1). We have first followed [13], Taylor-expanding (2.1) in powers of $1/s$. Inversion term by term gives the expansion of $f(l)$ near $l = 0$, and the resulting series is resummed by constructing rational Padé approximants of the form $lP(l^2)/Q(l^2)$, P and Q being polynomials of degree N and $N + 1$, respectively. For the 2D diffusion equation, we have extended the order of the Padé approximants to $N = 18$. The spin-flip ratios are generated using a rejection method, and the $\theta(x)$ can then be obtained. A comparison with the real spectra computed for the 2D diffusion equation is presented in figure 3.

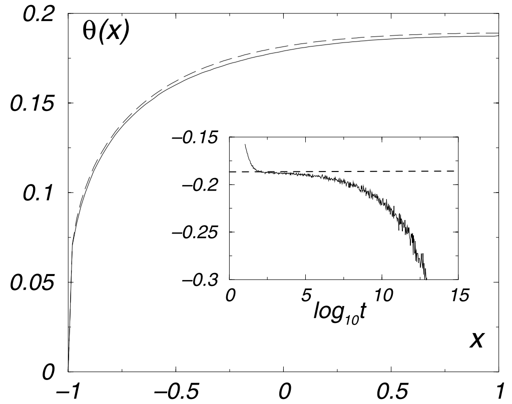

Figure 3. $\theta(x)$ as measured numerically for the 2D diffusion equation (lower, full curve, system size $11\,585^2$), and for a process with uncorrelated ratios of flip times, obtained via a rational Padé approximant of (2.1) (upper, broken curve). 500 values for x are considered in both cases. The inset shows the local slope of the generated $f(l)$ on a log–log scale, the horizontal broken line represents the theoretical value $\theta_{\mathrm{IIA}} \approx 0.1862$.

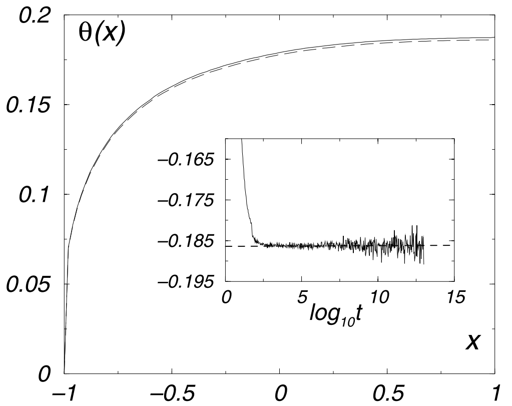

Figure 4. $\theta(x)$ as measured numerically for the 2D diffusion equation (upper, full curve, system size $11\,585^2$), and for a process (lower, broken curve) with uncorrelated ratios of flip times, of which the probability distribution function $f_Z(z) = f(l = \ln(1/z))|\mathrm{d}l/\mathrm{d}z|$ is tailored to match both the small- and large-l behaviours of the law $f(l)$ as encoded in (2.1). The inset shows the local slope of the generated $f(l)$ on a log–log scale, the horizontal broken line represents the theoretical value $\theta_{\mathrm{IIA}} \approx 0.1862$.

Over the whole range of the x-values, the agreement is excellent. Of course, beyond a certain range of times, the Padé approximants are no longer able to mimic the exponential decay of the probability distribution function $f(l)$ (see the inset of figure 3), which is the reason why the curve corresponding to the Padé method is systematically overshooting the real one.

To circumvent this limitation, we have eventually parametrized the probability distribution function $f(l)$ as $f(l) = (lU(l)/V(l)) \exp(-\theta l)$, U, V being polynomials in l of order N and $N + 1$, respectively. The coefficients of these polynomials are determined to match faithfully both the small-l ($f(l) = a_1 l + \cdots$) and large-l behaviour ($f(l) \approx a_0 \mathrm{e}^{-\theta l}$) of the law of the intervals. The implementation of this procedure (with $N = 4$) for the 2D diffusion equation leads to a discrepancy (figure 4) over the whole spectrum of $\theta(x)$ exponents which does not exceed the difference between—and is of the same sign as—the numerically measured value θ ($= \theta(1)$) ≈ 0.1875 and the value $\theta_{IIA} = 0.1862\ldots$ predicted by the IIA.

5. Conclusion

We have shown in this work how it is possible to gain a better understanding of persistence properties for processes having a finite density of zero crossings. The analytical results we have obtained, even though they have been derived within an approximate scheme, are in excellent agreement with numerical simulations conducted for the diffusion equation, which is considered as one of the simplest—yet non-trivial—model of coarsening. Furthermore, unexpected connections with relationships obtained in another class of stochastic processes have also emerged, which indicates that a certain universality of first-passage properties may exist in those systems, at least on a formal level. Let us also emphasize that, if one forgets the relationship (2.1) between the distribution function of intervals and the correlation function, it becomes possible [26] to account for all types of $\theta(x)$ spectra encountered in coarsening systems.

Finally, outside the realm of persistence and of its siblings, the method we have presented in section 3 should have a wide range of applicability for multiplicative stochastic processes. For instance, the calculation we have presented in section 3 for generalized persistence properties after a fixed number of spin-flips can be rephrased in terms of the exact solution of a random fragmentation problem, the latter topic being of ongoing interest [37–39]. It is also striking to note that first-passage exponents calculated recently for reaction–diffusion problems in the presence of disorder with the help of an asymptotically exact renormalization group also obey hypergeometric equations similar to (3.11) [40].

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